The Absolute Anabelian Geometry of Quasi-tripods

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k: a field of characteristic zero $\hookrightarrow \overline{k}$: an algebraic closure of k
p: a prime number

§1: Some Definitions

Definition

The field k is:

- algebraic $\overset{\text{def}}{\Leftrightarrow} k \overset{\exists}{\hookrightarrow} \overline{\mathbb{Q}}$
- sub-p-adic $\overset{\text{def}}{\Leftrightarrow} k \overset{\exists}{\hookrightarrow}$ $\exists$finitely generated/$\mathbb{Q}_p$
- generalized sub-p-adic $\overset{\text{def}}{\Leftrightarrow} k \overset{\exists}{\hookrightarrow}$ $\exists$fin. gen./the p-adic completion of $\mathbb{Q}_p^{\text{unr}}$
- strictly sub-p-adic $\overset{\text{def}}{\Leftrightarrow}$ sub-p-adic, $\mathbb{Q}_p \overset{\exists}{\hookrightarrow} k$

Thus:

$$\begin{array}{ccccc} \text{strictly sub-}p\text{-adic} & \Rightarrow & \text{sub-}p\text{-adic} & \Rightarrow & \text{generalized sub-}p\text{-adic} \\ & & \Uparrow & & \\ & & \text{finitely generated}/\mathbb{Q} & & \end{array}$$

Definition

X: a hyperbolic orbicurve/k $\overset{\text{def}}{\Leftrightarrow}$
$X \times_k K \cong$ a stack obtained by forming the quotient stack of
$\exists C$: a hyperbolic curve/K $\curvearrowleft$ $\exists G \subseteq \mathrm{Aut}_K(C)$
for $\exists K/k$: a finite extension

In particular, every hyperbolic curve is a hyperbolic orbicurve.

§2: Anabelian Problems

Relative Version

X, Y: varieties/k $\quad (\Rightarrow 1 \to \pi_1(X \times_k \overline{k}) \to \pi_1(X) \to G_k \to 1)$

$\Rightarrow$ Is $\mathrm{Isom}_k(X,Y) \to \mathrm{Isom}_{G_k}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y \times_k \overline{k}))$ bijective?

Theorem (Nakamura '90, Tamagawa '97, Mochizuki '03)

k: generalized sub-p-adic

X, Y: hyperbolic orbicurves/k

$\Rightarrow \mathrm{Isom}_k(X,Y) \to \mathrm{Isom}_{G_k}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y \times_k \overline{k}))$ is bijective.

Absolute Version

X, Y: varieties/k_X, k_Y: fields of characteristic zero, respectively

$\Rightarrow$ Is $\mathrm{Isom}(X,Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y))$ bijective?

Remark

If the basefields are fin. gen./$\mathbb{Q}$, then "Rel. Ver. $\Leftrightarrow$ Abs. Ver.". (Pop '95)

Theorem (Nakamura '90; Tamagawa '97; Mochizuki '96, '07, '12; Lepage)

One of the following holds:

- k_X, k_Y are fin. gen./$\mathbb{Q}$; X, Y are hyperbolic orbicurves.
- k_X, k_Y are finite/$\mathbb{Q}_p$; X, Y are "suitable" hyperbolic orbicurves.

$\Rightarrow \mathrm{Isom}(X,Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y))$ is bijective.

§3: Quasi-tripods

X: a hyperbolic orbicurve$/k$

Definition

Y: a hyperbolic orbicurve

$X \rightsquigarrow Y \quad \stackrel{\text{def}}{\Leftrightarrow} \quad$ one of:

- $X \stackrel{\text{finite étale}}{\to} Y \qquad$ • $Y \stackrel{\text{finite étale}}{\to} X \qquad$ • $X \stackrel{\text{open immersion}}{\hookrightarrow} Y$
- $X \to Y$ inducing $X_{\text{crs}} \stackrel{\sim}{\to} Y_{\text{crs}}$, where "$(-)_{\text{crs}}$" is the coarse scheme of "$(-)$"

Definition

X: a quasi-tripod$/k \stackrel{\text{def}}{\Leftrightarrow}$

$X \stackrel{\exists}{\rightsquigarrow} \exists X_1 \stackrel{\exists}{\rightsquigarrow} \dots \stackrel{\exists}{\rightsquigarrow} \exists X_n \stackrel{\exists}{\rightsquigarrow} \mathbb{P}^1_k \setminus \{0, 1, \infty\}$

Remark

$\forall$nonempty open substack of a quasi-tripod is a quasi-tripod

X: a hyperbolic curve $\Rightarrow$

- $\{\, U \subseteq X : \text{open} \mid U$ is a quasi-tripod$\,\}$ forms an open basis of X
- X is of genus $\leq 1 \Rightarrow X$ is a quasi-tripod

Suppose, moreover: k is finite$/\mathbb{Q}_p$, then

X is of Belyi-type $\Leftrightarrow$ X is a quasi-tripod and def'd/a fin. ext. of $\mathbb{Q}$

§4: Main Theorem

Main Theorem

k_X, k_Y: fields of characteristic zero
X, Y: hyperbolic orbicurves/k_X, k_Y, respectively

Assumption 1: Either X or Y is a quasi-tripod.

Assumption 2: One of the following holds:
(a) k_X and k_Y are algebraic, generalized sub-p-adic, and Hilbertian.
(b) k_X and k_Y are transc. and fin. gen./alg. and sub-p-adic subfields.
(c) k_X and k_Y are strictly sub-p-adic.

$\Rightarrow \mathrm{Isom}(X, Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y))$ is bijective.

Remark

The above theorem in the case where either
- k_X, k_Y are fin. gen./$\mathbb{Q}$ (Tamagawa '97) "$\subseteq$ (a) or (b)"

or
- k_X, k_Y are finite/$\mathbb{Q}_p$ (Mochizuki '07) "$\subseteq$ (c)"

was already proved.

§5: Application 1: Anabelian Open Basis

X: a smooth variety of positive dimension$/k$

Definition

X has a relatively (resp. an absolutely) anabelian open basis
$\overset{\text{def}}{\Leftrightarrow}$ $\exists \mathcal{U}$: an open basis of X s.t. for $\forall U,\ V \in \mathcal{U}$,
$\mathrm{Isom}_k(U, V) \to \mathrm{Isom}_{G_k}(\pi_1(U), \pi_1(V))/\mathrm{Inn}(\pi_1(V \times_k \overline{k}))$
(resp. $\mathrm{Isom}(U, V) \to \mathrm{Isom}(\pi_1(U), \pi_1(V))/\mathrm{Inn}(\pi_1(V)))$ is bijective.

Remark

A Prediction by Grothendieck (in a letter to Faltings)

k: finitely generated$/\mathbb{Q} \Rightarrow \exists$a rel. ($\overset{\text{cf. §2}}{\Leftrightarrow}$ abs.) anabelian open basis of X

(1) k: fin. gen.$/\mathbb{Q} \Rightarrow \exists$a rel. ($\overset{\text{cf. §2}}{\Leftrightarrow}$ abs.) anabelian open basis (Schmidt-Stix '16)
(2) k: generalized sub-p-adic $\Rightarrow \exists$a relatively anabelian open basis (H '20)
(3) k: finite$/\mathbb{Q}_p \Rightarrow \exists$an absolutely anabelian open basis (H '20)

Application

One of the following holds:
(a) k is algebraic, generalized sub-p-adic, and Hilbertian.
(b) k is transc. and fin. gen./an alg. and sub-p-adic subfield. "$\supseteq$ (1)"
(c) k is strictly sub-p-adic. "$\supseteq$ (3)"
$\Rightarrow \exists$an absolutely anabelian open basis of X

§6: Application 2: Configuration Spaces

Definition

X: a hyperbolic curve/k
n: a positive integer
X_n: the n-th configuration space of X, i.e.,

$$X_n \stackrel{\text{def}}{=} \overbrace{X \times_k \cdots \times_k X}^{n} \setminus \text{“}\{ (x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i \neq j \}\text{”}$$

Th. (Nakamura-Takao '98, Mochizuki-Tamagawa '08, Minamide-Mochizuki-H '17+)

k: generalized sub-p-adic
X, Y: hyperbolic curves/k
n_X, n_Y: positive integers
$\Rightarrow \mathrm{Isom}_k(X_{n_X}, Y_{n_Y}) \to \mathrm{Isom}_{G_k}(\pi_1(X_{n_X}), \pi_1(Y_{n_Y}))/\mathrm{Inn}(\pi_1(Y_{n_Y} \times_k \overline{k}))$ is bijective.

Application

k_X, k_Y: fields of characteristic zero
X, Y: hyperbolic curves/k_X, k_Y, respectively
n_X, n_Y: positive integers

Assumption 1: One of the following holds:
(1) Either X or Y is a quasi-tripod.
(2) $2 \leq \max\{n_X, n_Y\}$, and, moreover, either X or Y is affine.
(3) $3 \leq \max\{n_X, n_Y\}$.

Assumption 2: One of the following holds:
(a) k_X and k_Y are algebraic, generalized sub-p-adic, and Hilbertian.
(b) k_X and k_Y are transc. and fin. gen./alg. and sub-p-adic subfields.
(c) k_X and k_Y are strictly sub-p-adic.

$\Rightarrow \mathrm{Isom}(X_{n_X}, Y_{n_Y}) \to \mathrm{Isom}(\pi_1(X_{n_X}), \pi_1(Y_{n_Y}))/\mathrm{Inn}(\pi_1(Y_{n_Y}))$ is bijective.

§7-(1/5): Reconstruction Procedure

Recall:

Main Theorem

k_X, k_Y: fields of characteristic zero
X, Y: hyperbolic orbicurves/k_X, k_Y, respectively

Assumption 1: Either X or Y is a quasi-tripod.

Assumption 2: One of the following holds:
(a) k_X and k_Y are algebraic, generalized sub-p-adic, and Hilbertian.
(b) k_X and k_Y are transc. and fin. gen./alg. and sub-p-adic subfields.
(c) k_X and k_Y are strictly sub-p-adic.

$\Rightarrow$ $\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y))$ is bijective.

In the remainder, suppose:
- X is a quasi-tripod/k.

(b) k is transc. and fin. gen./an algebraic and sub-p-adic subfield.

§7-(2/5): Reconstruction Procedure

In the remainder, suppose:

- X is a quasi-tripod/k.

(b) k is transc. and fin. gen./an algebraic and sub-p-adic subfield.

$T_\square \stackrel{\text{def}}{=} \mathbb{P}^1_\square \setminus \{0, 1, \infty\}$
$F \subseteq k$: the maximal algebraic subfield of k $\quad$ ($\Rightarrow F$ is sub-p-adic (cf. (b)))
$\overline{F} \subseteq \overline{k}$: the alg. closure of F in $\overline{k}$ $\quad$ ($\Rightarrow \pi_1(X) \twoheadrightarrow G_k = \mathrm{Gal}(\overline{k}/k) \twoheadrightarrow G_F = \mathrm{Gal}(\overline{F}/F)$)

Rough Sketch of Procedure of Reconstruction

$\pi_1(X)$

$\stackrel{\text{(Step 1)}}{\Rightarrow}$ $\pi_1(X) \twoheadrightarrow G_k$ (by: Hilbertianity of k)

$\stackrel{\text{(Step 2)}}{\Rightarrow}$ $\pi_1(T_k) \twoheadrightarrow G_k$, hence $G_k \stackrel{\text{out}}{\curvearrowright} \pi_1(T_{\overline{k}})$ (by: group-theoreticity of "$\rightsquigarrow$")

$\stackrel{\text{(Step 3)}}{\Rightarrow}$ $\pi_1(T_k) \twoheadrightarrow G_k \twoheadrightarrow G_F$ (by: Belyĭ's faithfulness result)

$\stackrel{\text{(Step 4)}}{\Rightarrow}$ $\pi_1(T_F) \twoheadrightarrow G_F$ (by: center-free-ness of $\pi_1(T_{\overline{k}})$)

$\stackrel{\text{(Step 5)}}{\Rightarrow}$ $G_k \twoheadrightarrow G_F \curvearrowright \overline{F}$ (by: Belyi cuspidalization technique)

$\stackrel{\text{(Step 6)}}{\Rightarrow}$ $\pi_1(X) \twoheadrightarrow G_k \curvearrowright \overline{k}$ (by: Mochizuki/Pop's birational anabelian results)

$\stackrel{\text{(Step 7)}}{\Rightarrow}$ X (by: Mochizuki's anabelian results)

§7-(3/5): Reconstruction Procedure

Step 1

$\pi_1(X) \quad \Rightarrow \quad \pi_1(X) \twoheadrightarrow G_k$

Facts

- transcendental, finitely generated $\Rightarrow$ Hilbertian
- The absolute Galois group of a Hilbertian field has the following property:
  For a normal closed subgroup H
  H: top. fin. gen. (as an abstract topological group) $\Leftrightarrow H = \{1\}$
- π_1(a variety/an alg. cl. field of char. zero): topologically finitely generated

Thus, the kernel ($\cong \pi_1(X_{\overline{k}})$) of $\pi_1(X) \twoheadrightarrow G_k$ is
the unique maximal normal closed subgroup of $\pi_1(X)$
that is top. fin. gen. (as an abstract topological group).

Step 2

$\pi_1(X) \twoheadrightarrow G_k \quad \Rightarrow \quad \pi_1(T_k) \twoheadrightarrow G_k$, hence $G_k \overset{\text{out}}{\curvearrowright} \pi_1(T_{\overline{k}})$

Recall:

Definition

X: a quasi-tripod/k $\overset{\text{def}}{\Leftrightarrow}$
$X \overset{\exists}{\rightsquigarrow} \exists X_1 \overset{\exists}{\rightsquigarrow} \dots \overset{\exists}{\rightsquigarrow} \exists X_n \overset{\exists}{\rightsquigarrow} \mathbb{P}^1_k \setminus \{0, 1, \infty\}$

Fact: The "$\rightsquigarrow$" defined in

Definition

Y: a hyperbolic orbicurve
$X \rightsquigarrow Y \quad \overset{\text{def}}{\Leftrightarrow} \quad$ one of:

- $X \overset{\text{finite étale}}{\to} Y$ • $Y \overset{\text{finite étale}}{\to} X$ • $X \overset{\text{open immersion}}{\hookrightarrow} Y$
- $X \to Y$ inducing $X_{\text{crs}} \overset{\sim}{\to} Y_{\text{crs}}$, where "$(-)_{\text{crs}}$" is the coarse scheme of "$(-)$"

in §3 is "group-theoretic".

§7-(4/5): Reconstruction Procedure

Step 3

$\pi_1(T_k) \twoheadrightarrow G_k$, hence $G_k \overset{\text{out}}{\curvearrowright} \pi_1(T_{\overline{k}}) \quad \Rightarrow \quad \pi_1(T_k) \twoheadrightarrow G_k \twoheadrightarrow G_F$

By Belyĭ's faithfulness result,
the kernel of $G_k \to \mathrm{Out}(\pi_1(T_{\overline{k}}))$ = the kernel of $G_k \twoheadrightarrow G_F$.

Step 4

$\pi_1(T_k) \twoheadrightarrow G_k \twoheadrightarrow G_F \quad \Rightarrow \quad \pi_1(T_F) \twoheadrightarrow G_F$

By the center-free-ness of $\pi_1(T_{\overline{k}})$,
the resulting faithful outer action
$G_F \hookrightarrow \mathrm{Out}(\pi_1(T_{\overline{k}}))$ (cf. Step 3) determines

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(T_{\overline{k}}) & \longrightarrow & \mathrm{Aut} \times_{\mathrm{Out}} G_F & \longrightarrow & G_F & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(T_{\overline{k}}) & \longrightarrow & \mathrm{Aut}(\pi_1(T_{\overline{k}})) & \longrightarrow & \mathrm{Out}(\pi_1(T_{\overline{k}})) & \longrightarrow & 1.
\end{array}$$

Now it is easy to check that $\pi_1(T_F) \cong \mathrm{Aut} \times_{\mathrm{Out}} G_F$ over G_F.

§7-(5/5): Reconstruction Procedure

Step 5

$\pi_1(T_F) \twoheadrightarrow G_F \quad \Rightarrow \quad G_k \twoheadrightarrow G_F \curvearrowright \overline{F}$

Belyi cuspidalization technique

Substep 5-1

Fact: For $\forall a \in T_{\overline{F}}$: closed $\quad a \in \exists Z \overset{\text{closed}}{\subsetneq} T_{\overline{F}}$ and $T_{\overline{F}} \setminus Z \overset{\exists}{\to} T_{\overline{F}}$: finite étale

Point: C: a hyperbolic curve/F

$\pi_1(C) \Rightarrow$ the decomp. subgps of $\pi_1(C)$ ass'd to the cusps of C

(In general, $\pi_1(C) \overset{??}{\Rightarrow}$ the decomp. subgps of $\pi_1(C)$ ass'd to the closed points of C)

Thus, roughly speaking, by considering

- suitable open subgroups of $\pi_1(T)$, i.e., corr'g to "$\pi_1(T \setminus Z) \subseteq \pi_1(T)$",
- suitable quot.s of these op. subgps, i.e., corr'g to "$\pi_1(T \setminus Z) \twoheadrightarrow \pi_1(T \setminus \{a\})$",
- suitable quot.s of these quot.s, i.e., corr'g to "$\pi_1(T \setminus \{a\}) \twoheadrightarrow \pi_1(T)$", and
- the images of the "decomposition subgroups" w.r.t. these quotients, i.e., corr'g to "the image of $D_a \subseteq \pi_1(T \setminus \{a\}) \twoheadrightarrow \pi_1(T)$",

"$\Rightarrow$" $\{\pi_1(T_{F'} \setminus \{a\}) \twoheadrightarrow \pi_1(T_{F'}) \supseteq D_a\}_{F \overset{\text{finite}}{\subseteq} F' \subseteq \overline{F}:,\ a \in \mathbb{P}^1_{F'}(F')}$

$\Rightarrow$ the set $\overline{F}$ (cf. the fact that $\mathbb{P}^1_{F'}(F')$ "=" $F' \cup \{\infty\}$)

Substep 5-2

Facts

- $T = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ has an automorphism "$t \mapsto 1 - t$".
- $T \setminus \{a\} = \mathbb{P}^1 \setminus \{0, 1, \infty, a\}$ has an automorphism "$t \mapsto a/t$".

Thus, by applying

Theorem (Nakamura '90, Tamagawa '97, Mochizuki '03)

k: generalized sub-p-adic

X, Y: hyperbolic orbicurves/k

$\Rightarrow \mathrm{Isom}_k(X, Y) \to \mathrm{Isom}_{G_k}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y \times_k \overline{k}))$ is bijective.

in §2, together w/ some arguments,

$\Rightarrow$
- "$\overline{F} \ni x \mapsto ax \in \overline{F}$" by the automorphism "$t \mapsto a/t$"
- "$\overline{F} \times \overline{F} \ni (x, y) \mapsto x + y \in \overline{F}$" by the automorphism "$t \mapsto 1 - t$"

Step 6

$G_k \twoheadrightarrow G_F \curvearrowright \overline{F} \quad \Rightarrow \quad \pi_1(X) \twoheadrightarrow G_k \curvearrowright \overline{k}$

by Mochizuki/Pop's birational anabelian results

Step 7

$\pi_1(X) \twoheadrightarrow G_k \curvearrowright \overline{k} \quad \Rightarrow \quad X$

by Mochizuki's anabelian results

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